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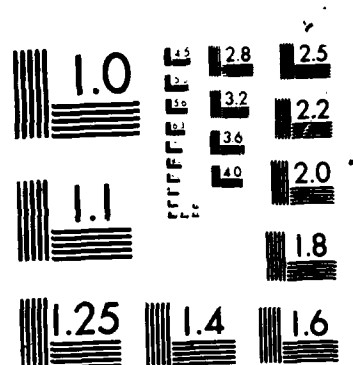
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Research under this grant focused on singular and bang-bang stochastic control and singular deterministic control. The investigators established the connections between a general Monotone Follower problem and a stopping problem: the spatial derivative of the value function of the Monotone Follower problem is the optimal risk function from the stopping problem. Similar results were obtained for the general Reflected Follower problem. This connection between singular control and optimal stopping has proved useful in establishing optimal stopping times via the corresponding control problem and permits application of analytical and numerical methods from optimal stopping to singular control problems. Nine papers were published under this grant, including "Connections between optimal stopping and singular stochastic control I", "Trivariate density of Brownian motion, its local and occupation time, with application to stochastic control", and "A stochastic control problem with different value functions for singular and absolutely continuous control."

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Optimal Control with Diminishing and Zero  
Cost for Control

Principal Investigators: S.E. Shreve and V.J. Mizel

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Optimal Control with Diminishing and Zero  
Cost for Control

Principal Investigators: S.E. Shreve and V.J. Mizel

Research under this grant focussed on singular and bang-bang stochastic control and singular deterministic control. In each of the following sections, we briefly outline these problems and describe progress made on them. The resulting publications are attached. The final section describes a book whose authorship was partially supported by this grant.

Section 1. Singular Stochastic Control

The first singular control models were those of spaceship control treated by Chernoff [10] and Bather and Chernoff [5,6]. Bather and Chernoff [5] cast their spaceship control problem in the following form:

$$\text{Minimize:} \quad E\left[\int_0^T \frac{1}{T-s} d\xi_s + \frac{1}{2} k x_T^2\right]$$

$$\text{Subject to:} \quad x_t = x + w_t - \xi_t + \zeta_t, \quad 0 \leq t \leq T.$$

In this formulation,  $\{w_t; 0 \leq t \leq T\}$  is a standard Brownian motion,  $\{\xi_t; 0 \leq t \leq T\}$  is a nondecreasing control process adapted to the filtration of the Brownian motion and satisfying

$\xi_0 = 0$ , and  $\{\xi_t; 0 \leq t \leq T\}$  is a nondecreasing adapted process which causes  $\{x_t; 0 \leq t \leq T\}$  to reflect at the origin, i.e.,

$$\xi_t = - \min_{0 \leq s \leq t} [(x - \xi_s + w_s) \wedge 0]; \quad 0 \leq t \leq T.$$

Such a singular control problem in which the state process is reflected at the origin will be called a Reflected Follower Problem. Bather and Chernoff conjectured that the optimal  $\xi$  should be the minimal process which keeps the state trajectory below a time-varying boundary  $f(t)$ , i.e.,

$$\xi_t = \max_{0 \leq s \leq t} [(x + w_t + \xi_t - f(t)) \vee 0]; \quad 0 \leq t \leq T.$$

One then would expect the value function for this problem to satisfy the Hamilton-Jacobi-Bellman equation

$$V_t(t, x) + \frac{1}{2} V_{xx}(t, x) = 0, \quad 0 \leq x \leq f(t), \quad 0 \leq t \leq T,$$

with boundary conditions

$$\text{Reflection at origin: } V_x(t, 0) = 0; \quad 0 \leq t \leq T$$

$$\text{Terminal cost: } V(T, x) = \frac{1}{2} kx^2; \quad x \geq 0.$$

On the boundary  $\{(t, x); x = f(t), 0 \leq t \leq T\}$ , the marginal cost of pushing should equal the marginal decrease in value, i.e.,

$$\text{Boundary: } V_x(t, f(t)) = \frac{1}{T-t}; \quad 0 \leq t \leq T.$$

Setting  $R(t, x) = V_x(t, x)$ , Bather and Chernoff then derived the conditions

$$R_t(t, x) + \frac{1}{2} R_{xx}(t, x) = 0; \quad 0 \leq x \leq f(t), \quad 0 \leq t \leq T.$$

$$R(t, 0) = 0; \quad 0 \leq t \leq T.$$

$$R(T, x) = kx; \quad x \geq 0.$$

$$R(t, x) = \frac{1}{T-t}; \quad 0 \leq t \leq T,$$

and recognized these as the free boundary conditions associated with the problem of stopping a Brownian motion so as to minimize the expected sum of (a) zero if the motion reaches the origin before terminal time  $T$  and before it is stopped, (b)  $kx$  if the motion does not reach the origin and is not stopped before terminal time  $T$ , and arrives then at position  $x$ , (c)  $\frac{1}{T-t}$  if the motion is stopped at time  $t < T$ . They then used this optimal stopping problem to provide bounds on the free boundary  $f$ .

In their seminal 1980 paper [9], Benes, Shepp and Witsenhausen posed two singular control problems. One of them was the following Monotone Follower Problem:

$$\text{Minimize: } E \int_0^T (x + w_t - \xi_t)^2,$$

where  $\xi$  and  $w$  are as in the previous problem. They found a

boundary of the form  $f(t) = \delta\sqrt{T-t}$  such that the optimal  $\xi$  is the minimal process which keeps the state trajectory  $x + w_t - \xi_t$  below  $f(t)$ , i.e.,

$$\xi_t = \max_{0 \leq s \leq t} [(x + w_s - f(s)) \vee 0]; \quad 0 \leq t \leq T.$$

They also noted that this boundary is of the same form as that found by Miroshnichenko [21] to demarcate the stopping and continuation regions for the problem

$$\text{Minimize:} \quad E \int_0^T (x + w_t) dt.$$

where the minimum is over stopping times  $\tau$  taking values in  $[0, T]$ . Benes, et al., also pointed out the connection with local time; namely, the optimal  $\xi$  is, except for a possible initial jump, equal to the local time of the state process on the boundary  $\{(t, x): x = f(t), 0 \leq t \leq T\}$ :

$$\xi_t - \xi_{0+} = \lim_{\epsilon \downarrow 0} \frac{1}{2\epsilon} \int_0^t 1[f(s) - \epsilon, f(s)] (x + w_s - \xi_s) ds.$$

Motivated by these broad hints that some connection existed between the optimal stopping and singular control, we used this grant to support our investigation of this issue. In Karatzas and Shreve [18], we established by probabilistic means the



following results. For the general Monoton Follower Problem

$$\begin{aligned} \text{Minimize:} \quad & E \left[ \int_0^T h(t, x_t) dt + \int_{[0, T]} f(t) d\xi_t + g(x_T) \right] \\ \text{Subejct to:} \quad & x_t = x + w_t - \xi_t; \quad 0 \leq t \leq T, \end{aligned}$$

where  $\xi$  is nondecreasing, the spatial derivative of the value function is the optimal risk function for the stopping problem

$$\text{Minimize:} \quad E \left[ \int_0^T h_x(t, x+w_t) dt + f(\sigma) 1_{\{\tau < T\}} + g'(x+w_t) 1_{\{\tau = T\}} \right],$$

where the minimum is over all stopping times  $\tau$  lying in  $[0, T]$ . This result was proved under the assumptions that  $f, g_x$  and  $h_x$  are continuous,  $g$  and  $h$  are convex in  $x$  and grow at most polynomially, and there exists an optimal process for the control problem. Furthermore, the region of the state space in which no control is exercised in the control problem (the optimal  $\xi$  is constant) is the optimal continuation region in the stopping problem. In particular, an optimal stopping time exists in the stopping problem. It was also shown that if  $f(t)$  is positive and bounded away from zero,  $h$  and  $g$  are nonnegative, and  $\sup_{x \in \mathbb{R}} g'(x) \leq f(T)$ , then an optimal process exists for the control problem.

In Karatzas and Shreve [20] we also established similar results for a general Reflected Follower Problem. The principle

new feature is that for these problems, the Brownian motion in the associated stopping problem is absorbed at the origin, a fact already observed by Bather and Chernoff [5].

One benefit of the established connection between singular control and optimal stopping is that bounds on the continuation region of the stopping problem, obtained by posing and solving more and less favorable problems, translate into bounds on the region of inaction in the control problem. This comparison idea goes back to Bather [4] and has been used by Bather and Chernoff [5], Chernoff [10], and Karatzas [17]. A second benefit of the connection between these problems is that, because control processes are more easily topologized than stopping times and are thus more amenable to the continuity and compactness arguments frequently used in existence proofs, it is feasible to prove the existence of optimal stopping times via the corresponding control problem. Such a line of argument was used in [18], and it avoids the heavy regularity assumptions usually required to prove the existence of optimal stopping times. See, for example, Friedman [12]. A third benefit flows in the other direction: from stopping to control. Because optimal stopping is a mature research area, as compared to singular control, there are a large number of analytical and numerical methods which can be brought via optimal stopping to bear on singular control. In particular, the theory of numerical solution of Stefan problems becomes available.

## Section 2. Bang-Bang Stochastic Control

One of the first uses of Girsanov's formula to solve a stochastic control problem is due to Benes [7]. This so-called "predicted miss" problem is to minimize the distance of the final state  $x(1) \in \mathbb{R}^d$  from a hyperplane subject to the system equation

$$x_t = x + \int_0^t A(s)x_s ds + \int_0^t B(s)u(s, x_s)ds + \int_0^t C(s)dw_{\sim s}.$$

where  $A$ ,  $B$  and  $C$  are deterministic matrix functions of time,  $w$  is a multi-dimensional Wiener process, and the control is constrained by  $u: [0,1] \times \mathbb{R}^d \rightarrow [-1,1]^r$ . Davis and Clark [11] subsequently showed how to reduce this multi-dimensional problem to the one-dimensional problem:

$$\text{Minimize: } Ex^2(1)$$

Subject to

$$x_t = x + \int_0^t u(s, x_s)ds + w(t),$$

$$|u(t, x_t)| \leq 1,$$

where  $w$  is a one-dimensional Wiener process. This one-dimensional problem has been solved by different methods by Benes [8], Ikeda and Watanabe [16], Davis and Clark [11], Haussman [13], Balakrishnan [1] and Shreve [24]. The obvious

control law  $u(t, x_t) = -\text{sgn}(x_t)$  is indeed the optimal one. Shreve [24] computed the transition probabilities for the controlled process

$$(2.1) \quad x_t = x - \int_0^t \text{sgn}(x_s) ds + w_t$$

using Girsanov's formula and local time in a way very similar to the method we are about to describe for the asymmetric equation (2.2).

Benes, Shepp and Witsenhausen [9] solved the related problem:

$$\text{Minimize:} \quad E \int_0^\infty e^{-\alpha t} x_t^2 dt,$$

Subject to:

$$x_t = x + \int_0^t u(s, x_s) ds + w_t$$

$$-\infty < \theta_0 \leq u(s, x_s) \leq \theta_1 < \infty.$$

They proved the existence of a number  $\delta$  such that an optimal control law is given by

$$u(t, x_t) = \begin{cases} \theta_0 & \text{if } x_t < \delta. \\ \theta_1 & \text{if } x_t \geq \delta. \end{cases}$$

The resulting controlled process is governed by the equation

$$(2.2) \quad x_t = x + \int_0^t \theta(x_s) ds + w_t,$$

where

$$\theta(x) = \begin{cases} \theta_0 & \text{if } x < \delta, \\ \theta_1 & \text{if } x \geq \delta. \end{cases}$$

This is an asymmetric version of (2.1).

In the work supported by this grant, which appears in Karatzas and Shreve [19], we computed the transition density for  $x_t$  governed by (2.2) as follows. According to Girsanov's formula, the desired density is given by

$$(2.3) \quad P^x[x_t \in dy] = E^x[\zeta(t) 1_{\{w_t \in dy\}}],$$

where

$$\zeta(t) = \exp\left[\int_0^t \theta(w_u) dw_u - \frac{1}{2} \int_0^t \theta^2(w_u) du\right].$$

Our computation thus requires knowledge of the joint distribution

$$\text{of } (w(t), \int_0^t \theta(w_u) dw_u, \int_0^t \theta^2(w_u) du).$$

Assuming for notational simplicity that  $\theta_0 = 0$ .

$$(2.4) \quad \int_0^t \theta^2(w_u) du = \theta_1 \Gamma(t) + \theta_0 (t - \Gamma(t))$$

where

$$\Gamma(t) = \int_0^t 1_{\{w_s \geq 0\}} ds$$

is the occupation time for the right half-line. A variation of Tanaka's formula gives

$$(2.5) \quad \int_0^t \theta(w_u) dw_u = \phi(w_t) - \phi(w_0) + (\theta_1 - \theta_0) L(t),$$

where

$$\phi(z) = \int_0^z \theta(y) dy$$

and  $L(t)$  is the local time of  $w$  at the origin:

$$L(t) = \lim_{\epsilon \downarrow 0} \frac{1}{4\epsilon} \int_0^t 1_{\{-\epsilon < w_s < \epsilon\}} ds.$$

The local time appears because of the discontinuity of  $\theta$  at  $\delta$

$= 0$ . Thus, our problem has been reduced to a computation of the trivariate density  $P^x[w_t \in da, L(t) \in db, \Gamma(t) \in dr]$  of Brownian motion, its local time at the origin, and its occupation time of the right half-line. Karatzas and Shreve [19] contains a derivation via Laplace transforms and the Feynman-Kac formula for elastic Brownian motion, and this paper also sketches a proof based on a formula of D. Williams [25] involving inverse local times.

### Section 3. Singular Deterministic Control

This grant supported an investigation into the foundations of the calculus of variations which led to the discovery that even for regular one-dimensional control problems of Lagrange type the optimal solutions ("minimizers") can exhibit several remarkable features. For one thing, when the asymptotic behavior of running cost of control is markedly different at certain states from its behavior at others, then the optimal solution may have infinite slope at certain times and can fail to satisfy the integrated form of the Pontryagin minimum principle.

Particularly striking is the Lavrentiev phenomenon: under the above circumstances, the optimal solution may be such that the standard methods for approximating optimal solutions of control problems (penalty methods, finite element methods and the like) will all converge not to the minimizer itself, but instead to a higher cost "pseudominimizer". It follows that these methods cannot in general detect singular minimizers, and will instead lead to overly pessimistic cost estimates and suboptimal control

policies (both of which are not subject to improvement with increasing computational accuracy). This work is reported in Ball and Mizel [2,3].

The generalization of the model studied to allow for a noise term leads to a singular stochastic control problem of the type discussed in Section 1. The Lavrentiev phenomenon takes the form of a gap between the best performance with a non-singular control policy and the best performance with a singular control policy. This work was the basis for Heinricher [14] and is succinctly reported in Heinricher and Mizel [15].

This grant also supported Mizel and Trutzer [22,23] for work on stochastic hereditary equations.

#### Section 4. Martingales, Brownian Motion and Stochastic Differential Equations

During the work described in this proposal, we have found the difficult and esoteric concept of Brownian local time to be an indispensable tool in dealing with a number of stochastic control problems. To make local time a bit less esoteric, we have undertaken to write a book on its general nature motivated by its applications to control. This book will present many of the problems discussed here, and, so as to be nearly self-contained, it will also include a full treatment of Brownian motion, stochastic integration with respect to continuous, square-integrable martingales, and strong and weak solutions of stochastic differential equations. It is our goal to make the topics treated in this proposal accessible to the matheamtics.



engineering and economics communities.

Upon the expiration of this grant, the book was about half completed. A table of contents for the entire book is provided.

MARTINGALES, BROWNIAN MOTION AND STOCHASTIC  
DIFFERENTIAL EQUATIONS

by

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- 6.8: References

Section 5. References (\* indicates publications supported by this grant.)

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## Research on: Optimal Control with Diminishing or Zero Cost of Control.

In studying the effect, in deterministic control problems, of very marked dependence on system state, of the asymptotic growth of cost of control, we are preparing the ground for application of control to relatively sophisticated systems. In the long run we hope to analyze control properties for multidimensional nonlinear systems such as those occurring in continuous elastic structures stressed to configurations near their yield points. It is well known that such systems are subject to "fracture", and one outgrowth of the present work may be an understanding of whether fracture can arise in consequence of the minimizer for the energy integral, although continuous, having a gradient which is infinite along a low dimensional portion of the given structure. In the near term we expect to further clarify the one dimensional case through the use of classical analysis on spaces of absolutely continuous functions.

Our work with Professor John Ball seems to have been the first to demonstrate regular problems with such anomalies as failure of the minimizer to satisfy the integrated Euler-Lagrange and "energy" equations (i.e., the maximum principle), as well as the existence of a non zero gap between the actual minimum value of the cost integral and the lower limit it assumes for continuous controls ("Lavrentiev phenomenon"). Other researchers are currently becoming involved in related issues involving the limitations of classical control and variational theories. The Lavrentiev phenomenon has important implications for the numerical solution of control and variational problems since numerical approximation inevitably involves smooth admissible functions. An investigation is under way to analyze the behavior of corresponding diffusion type stochastic control problems (in which the cost of additional control has similar asymptotic growth properties), in order to test whether the limit of small noise perturbation will provide more easily treated approximation methods for the deterministic problem.

The long range purpose of our work in stochastic control is to expand the set of models which are well understood analytically. Except for the linear-quadratic-Gaussian regulator and the models presented by Beneš, Shepp and Witsenhausen in 1980, there are very few stochastic control problems which can be "solved" in any sense other than numerically. We have attacked specific problems in "bang-bang" and singular stochastic control; the former have arisen in guidance and the latter in inventory and queue control. Our work in bang-bang control uses the tools of Brownian local time and path decomposition, both of which are now under active investigation by probabilists. The singular control problems date back to work by Chernoff in 1968, but are only now receiving widespread attention.

In the next year we expect to learn more about local times and path decompositions, and we also expect to solve other specific control problems using these techniques. In contrast to singular control, which now benefits from much activity, we are aware of no other current work in the important and interesting area of bang-bang control, so we expect our emphasis to switch somewhat in that direction.

Recent Developments in Deterministic and  
Stochastic Optimal Control (AFOSR-82-0259)

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February 8, 1984

During the past year, our investigation of the influence, in deterministic control problems, of varying the growth rate for cost of control has led to two significant developments. First, it has been demonstrated that even for regular one-dimensional Lagrange type control problems (with polynomial integrand), under growth conditions which ensure existence of a global absolutely continuous minimizer, one can have a global minimizer which fails totally to satisfy the integrated form of the maximum principle! More specifically, examples have been constructed of calculus of variations problems in one real function whose global minimizer satisfies neither the integrated form of the Euler-Lagrange equations nor the integrated form of the DuBois-Reymond ("energy") equation. These examples are apparently the first known uniformly regular problems possessing absolutely continuous but non-Lipschitz minimizers. The feature of the integrand which permits such an anomaly is that, despite a uniformly quadratic lower growth in the cost of control, the polynomial growth of cost of control is affected in an essential way by the choice of state at which control is applied.

Second, it was found that the long forgotten "Lavrentiev phenomenon" (discussed in the recent book of Cesari) can readily occur in such problems. That is, one can have a problem with a

non-Lipschitz minimizer such that Lipschitz functions cannot simultaneously approximate the minimizer itself and the cost incurred with this minimizer; the infimum of the costs attained by Lipschitz trajectories can be strictly greater than the cost achieved by the absolutely continuous minimizer! The implications of this phenomenon for the numerical study of control problems are profound--numerically generated costs are inevitably connected with Lipschitz trajectories. Hence the numerical study of such control problems will invariably lead to overly pessimistic conclusions concerning their optimal cost unless techniques are consciously followed to bypass the Lavrentiev phenomenon.

The objective of our work in stochastic control has been to study problems in which control is exercised in a "bang-bang" fashion, (i.e., it switches discontinuously as the state process crosses a "switching surface"), or in which control is singular, (i.e., it can be thought of as the limit of impulse controls as the sizes of the impulse approaches  $\infty$  while the duration approaches zero). The former problems first arose in the context of rocket guidance with the desired trajectory serving as a switching surface; the latter problems appear in inventory control where demand and supply processes provide impulses to the inventory process. We have thus far succeeded in computing the statistics of bang-bang controlled processes in some interesting one-dimensional cases, and we believe the collateral discoveries we have made about Brownian local time and path decompositions are of independent interest. We have also shown how to reduce some fairly large classes of singular control problems to much studied problems in optimal stopping. We expect this reduction to result in interplay between the two kinds of problems so as to enhance understanding of both.



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